ASSIGNMENT SET – I

Mathematics: Semester-IV

M.Sc (CBCS)

Department of Mathematics

Mugberia Gangadhar Mahavidyalaya

PAPER - MTM-401

Paper: Functional Analysis

1. What is Hilbert-adjoint operator? Show that the Hilbert-adjoint operator T^* of $: H_1 \rightarrow H_2$, where H_1 and H_2 are Hilbert spaces exists, unique and is a bounded linear operator with norm $\|T^*\| = \|T\|$.
2. If $\{e_1, e_2, e_3, \dots, e_n\}$ is a finite orthonormal set in an inner product space X and x is any element of X, then prove that $\sum_{i=1}^{n} |\langle x,\ e_j \rangle|^2 \leq ||x||^2$. When equality holds?
3. Prove that inner product is continuous.
4. Show that a subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.
5. Show that the dual space of a normed space is a Banach space.
6. Give the definition of Closed Linear Operator in sequential form. State and prove Open Mapping Theorem.
7. Show that the space $l^p$is complete; here p is fixed and $1 \leq p < +\infty$.
8. Give an example of an incomplete metric space and justify your answer.
9. Show that a Hilbert space $H$is separable if every orthonormal set in H is countable.
10. Discuss strong convergence and weak convergence. State and prove Uniform Boundedness principle.

11. State and prove Hann-Banach Extension theorem. Define Quotient space with norm.
12. Let X and Y be inner product spaces. Then a linear map $F: X \to Y$satisfies $\langle F(x), F(y)\rangle = \langle x, y\rangle$ for all $x, y \in X$ if and only if it satisfies $\|F(x)\| = \|x\|$ for all $x \in X$, where the norms on X and Y are induced by the respective inner products.
13. Let $P \in BL(\mathcal{H})$ be a nonzero projection on a Hilbert space $\mathcal{H}$and $\|P\| = 1$. Then show that P is an orthogonal projection.
14. Let the space $l^2(\mathbb{Z})$be defined as the space of all two- sided square summable sequences and the bilateral shift is the operator Won $l^2(\mathbb{Z})$ defined by $W(\dots, a_{-2}, a_{-1}, \hat{a}_0, a_1, a_2, \dots) = (\dots, a_{-3}, a_{-2}, \hat{a}_{-1}, a_0, a_1, \dots)$. Prove that

(i)W is unitary, and

(ii) the adjoint W^* of W is given by $W^*(\dots, a_{-2}, a_{-1}, \hat{a}_0, a_1, a_2, \dots) = (\dots, a_{-1}, a_0, \hat{a}_1, a_2, a_3, \dots)$.

15. Let X and Y be two normed linear spaces over the same field of scalars and let $T : X \to Y$ be a linear operator that sends a convergent sequence in X to a bounded sequence in Y. Prove that T is a bounded linear operator.
16. Let X and Y be Banach spaces and $A \in BL(X, Y)$. Show that there is a constant $c > 0$ such that $\| Ax \| \geq c \| x \|$ for all $x \in X$ if and only if $Ker(A) = \{0\}$ and $Ran(A)$ is closed in X.
17. Let $T : C[0,1] \to C[0,1]$ be defined by $T(t) = \int_0^t x(\tau)d\tau$. Find $R(T)$ and obtain $T^{-1}: R(T) \to C[0, 1]$. Examine if T^{-1} is linear and bounded.
18. Show that $\langle Ae_j , e_i \rangle = (i + j + 1)^{-1}$ for $0 \leq i, j \leq \infty$ defines a bounded operator $l^2(\mathbb{N} \cup \{0\})$ with $\|A\| \leq \pi$.
19. Let $S \in BL(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space. Prove that for all $x, y \in \mathcal{H}$, $\langle Sx, y\rangle = \frac{1}{4}\sum_{n=0}^{3} i^n \langle S(x + i^n y), (x + i^n y) \rangle$.
20. Check whether C^1 [0, 1] with the supremum norm is a Banach space
21. Show that the space l^p is complete, here p is fixed and $1 \leq p \leq \infty$.
22. Let $T: X \to Y$ be a continuous linear operator. Show that the Null space $\mathcal{N}(T)$ is closed.

23. Prove that a mapping T of a metric space X into a Metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X.

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